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TECHNICAL NOTE

Surface settlements due to deformation of a tunnel in an elastic half plane

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KEYWORDS: deformation; elasticity; stress analysis; tunnels.

INTRODUCTION

Deformations of a tunnel may result in settlements of the ground surface. In engineering practice these surface settlements are often described by empirical formulae, based upon field observations, for instance a normal (Gaussian) distribution curve (Peck, 1969; Attewell & Woodman, 1982). In this note an approximate analytical solution is presented, considering the soil as a linear elastic material. Although it is realized that this is a very poor schematization of the real behaviour of soils, an elastic solution may well serve to investigate some of the characteristics of the resulting fields of stress and strain, and it may also serve as a reference for more refined (numerical) computations.

Two basic deformation mechanisms of the tunnel are considered: a uniform radial displacement (representing, in first approximation, the ground loss that may occur during construction of the tunnel), and an ovalization of the tunnel, see Fig. 1. Even though in practice the tunnelling process may be executed very carefully and appropriate engineering techniques may be applied to minimize the deformations, for instance the injection of grout into the soil surrounding the tunnel, it remains of interest to study the deformations and stresses caused by both ground loss and ovalization.

The method used is an extension of a method suggested by Sagaseta (1987) for the case of ground loss in an incompressible soil. The starting point is the solution for a singularity at a point of an elastic half plane (at the axis of the tunnel). By adding the image solution for a singularity at a point located symmetrically above the soil surface, the shear stresses at the surface are made equal to zero. In order to balance the normal stresses at the surface, a third solution is added, by solving a Boussinesq-type problem. The approximation in establishing the solution is that, in balancing the normal and tangential stresses on the surface, the presence of the tunnel, with its prescribed displacements, is disregarded. Although an exact solution might well be obtainable by the methods outlined by Mindlin (1939), using bipolar coordinates, such a solution is very complicated. The present approximate solution will be seen to be very simple and easy to apply.

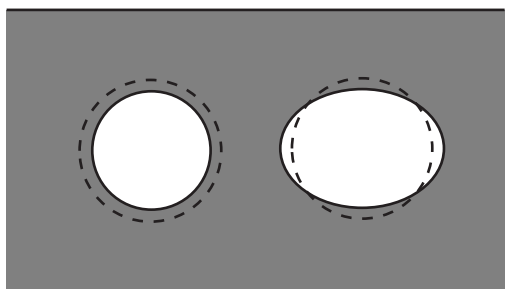

Fig. 1. Ground loss and ovalization of a tunnel

The solution given in this technical note is a generalization of Sagaseta's solution in that it gives the solution for the case of ground loss not only for the incompressible case (with Poisson's ratio equal to 0·5), but for arbitrary values of Poisson's ratio, and that it includes the effect of ovalization. The process of ovalization has been studied before for a tunnel in an infinite elastic medium (Muir Wood, 1975). Here the surface displacements for a tunnel in a semi-infinite medium are considered, as well as the displacements and stresses throughout the half space.

THE SOLUTION

The (approximate) solution of the problem is supposed to consist of three parts. The first two

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parts are singular solutions from the theory of elasticity in the points $x = 0$, $z = h$ and $x = 0$, $z = -h$, (Fig. 2). In these solutions the displacements are of order $O(1/r)$, so that they correspond to the removal of a finite volume, and the stresses are of order $O(1/r^2)$. They are well-known basic solutions from the theory of elasticity (Timoshenko & Goodier, 1951). All other possible singular solutions are of larger negative orders of r, and thus will be small compared to the present one if the radius R of the tunnel is small enough.

Taken together, the expressions for the displacement components u_x and u_z for the singular parts of the solution and their images are

$$u_x = -\varepsilon R^2\left(\frac{x}{r_1^2}+\frac{x}{r_2^2}\right) + \delta R^2\left(\frac{x(x^2-kz_1^2)}{r_1^4}+\frac{x(x^2-kz_2^2)}{r_2^4}\right) \quad (1)$$

$$u_z = -\varepsilon R^2\left(\frac{z_1}{r_1^2}+\frac{z_2}{r_2^2}\right) + \delta R^2\left(\frac{z_1(kx^2-z_1^2)}{r_1^4}+\frac{z_2(kx^2-z_2^2)}{r_2^4}\right) \quad (2)$$

where ε and δ are parameters indicating the relative displacement of the tunnel surface, for the uniform radial displacement case (ε) and the ovalization case (δ) respectively. In these solutions $z_1 = z - h$, $z_2 = z + h$, and r_1 and r_2 are the distances from the singular point and its image (Fig. 2). It can easily be verified that these solutions satisfy the basic differential equations from the theory of plane strain elasticity.

Because of the symmetry of these two solutions, the shear stress σ_{zx} and the vertical displacement u_z will be zero at the surface $z = 0$. The normal stresses σ_{zz} induced by the two singular solutions are equal, and add up to

$$z = 0 : \sigma_{zz} = q(x) = -4\mu\varepsilon R^2\frac{x^2-h^2}{(x^2+h^2)^2} - \frac{8\mu m\delta R^2}{m+1}\frac{h^2(3x^2-h^2)}{(x^2+h^2)^3} \quad (3)$$

where μ is the shear modulus of the elastic material, and m is an auxiliary elastic constant, related to Poisson's ratio by

$$m = \frac{1}{1-2\nu} \quad (4)$$

In order to satisfy the boundary condition that the normal stress $\sigma_{zz} = 0$ at the surface $z = 0$, a third solution must be added, which must balance the stress distribution $q(x)$. This third part of the solution is the solution of a Boussinesq problem, for the half plane $z > 0$, with the boundary conditions

$$z = 0 : \sigma_{zz} = -q(x),\ \sigma_{zx} = 0. \quad (5)$$

Problems of this type can be solved most conveniently by using Fourier transforms (Sneddon, 1951). It can be shown that, if the load function is even, $q(-x) = q(x)$, the solution of the problem with the boundary conditions (5) can be written as

$$u_x = \int_0^\infty C(1-m\alpha z)\exp(-\alpha z)\sin(\alpha x)\,\mathrm{d}\alpha \quad (6)$$

$$u_z = -\int_0^\infty C(1+m+m\alpha z)\exp(-\alpha z)\cos(\alpha x)\,\mathrm{d}\alpha \quad (7)$$

where C is defined by the integral

$$C = -\frac{1}{m\pi\mu}\frac{1}{\alpha}\int_0^\infty q(x)\cos(\alpha x)\,\mathrm{d}x \quad (8)$$

In this case, with the function $q(x)$ defined by (3) the function C is found to be

$$C = -\frac{2\varepsilon R^2}{m}\exp(-\alpha h) - \frac{2\delta R^2}{m+1}\alpha h\exp(-\alpha h) \quad (9)$$

Substitution of this result into equations (6) and (7) gives, after evaluation of the Fourier integrals, using known integrals from Bateman (1954) and some results that can easily be derived from them,

$$u_x = -\frac{2\varepsilon R^2 x}{m}\left(\frac{1}{r_2^2}-\frac{2mzz_2}{r_2^4}\right) - \frac{4\delta R^2 xh}{m+1}\left(\frac{z_2}{r_2^4}+\frac{mz(x^2-3z_2^2)}{r_2^6}\right) \quad (10)$$

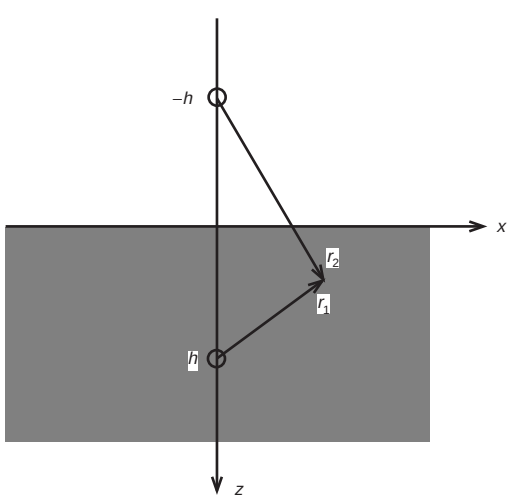

Fig. 2. A singularity and its image

$$u_z = \frac{2\varepsilon R^2}{m}\left(\frac{(m+1)z_2}{r_2^2} - \frac{mz(x^2 - z_2^2)}{r_2^4}\right) - 2\delta R^2 h\left(\frac{x^2 - z_2^2}{r_2^4} + \frac{m}{m+1}\frac{2zz_2(3x^2 - z_2^2)}{r_2^6}\right) \tag{11}$$

The complete solution of the problem is the sum of this solution and the expressions (1) and (2).

All stresses and strains can be directly derived from the solution given above. The vertical displacement of the surface $z = 0$, denoted as u_0, is of particular interest. Taking $z = 0$ in equation (11) gives

$$u_0 = 2\varepsilon R^2 \frac{m+1}{m}\frac{h}{x^2 + h^2} - 2\delta R^2 \frac{h(x^2 - h^2)}{(x^2 + h^2)^2} \tag{12}$$

This is the total settlement of the surface, because for $z = 0$ the two singular solutions in equation (2) cancel.

The shape of the two settlement functions is shown in Fig. 3. The solid line is the settlement curve for the first term, due to the ground loss. The dashed line is the settlement curve for the second term, due to the ovalization of the tunnel. It appears that the width of the settlement trough is considerably smaller for the ovalization case than for the ground loss case. This may be one explanation for the rather narrow settlement troughs usually observed in tunnelling practice. Another explanation may of course be that the soil behaviour is non-linear, exhibiting plastic effects.

The factor $(m + 1)/m$ in equation (12) can also be written as $2(1 - \nu)$. For $\nu = 0{\cdot}5$ this factor is 1. For that value of ν the first term of the solution has been presented earlier by Sagaseta (1987).

The total area (A) of the settlement trough is found by integrating equation (12) from $-\infty$ to $+\infty$. The result is

$$A = \frac{m+1}{m} 2\varepsilon\pi R^2 = 4(1 - \nu)\varepsilon\pi R^2 \tag{13}$$

For $\nu = 0{\cdot}5$ this is precisely the ground loss, $2\varepsilon\pi R^2$, a result also mentioned by Sagaseta (1987). It is interesting to note that for other values of ν the total area below the settlement curve is larger than the ground loss. This initially somewhat surprising result must be attributed to the deformations at infinity. For $\nu = 0$ the total area is double that for an incompressible material, and thus Sagaseta's solution will in general lead to an underestimation of the settlement under drained conditions. It may also be concluded that it is of great practical importance to keep the ground loss as small as possible, in order to prevent surface settlements, a fact long recognized in tunnel construction.

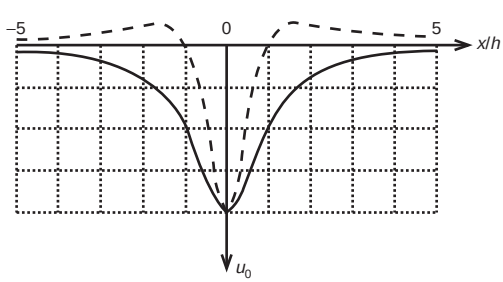

Fig. 3. Surface settlements

The shape of the settlement curve is independent of ν, but its values increase if ν decreases from 0·5 to 0. In a consolidating, porous, elastic medium this would mean that the immediate settlement is equal to the ground loss, with the consolidation settlement leading to a gradual but approximately conformal increase of the settlements. This will be the subject of a subsequent publication.

It may also be interesting to note that the average value of the ovalization term in the solution is zero. In the zone $x < h$ there is a settlement, but in the zone $x > h$ there is some heave, and the two of these appear to balance on average.

In order to assess the accuracy of the approximation, the total displacements at the tunnel's radial circumference may be determined. The first term of the solution represents the constant radial displacement due to the singularity in an infinite medium. The other two solutions disturb this displacement, and the relative magnitude of this disturbance is a measure of the accuracy of the solution. This can be evaluated by comparing the radial displacements at the tunnel face due to the second and the third parts of the solution with those due to the first part. It appears that the error is less than 10%, provided that the radius of the tunnel is less than one half of its depth.

STRESSES

The strain components can easily be derived by differentiation from the expressions for the displacements, and then the stress components can be obtained with the aid of Hooke's law. The isotropic stress $\sigma_0 = \frac{1}{2}(\sigma_{xx} + \sigma_{zz})$ is found to be, for instance,

$$\sigma_0 = -4\mu\varepsilon R^2 \frac{z_2^2 - x^2}{r_2^4} + \frac{\delta R^2}{1 - \nu}\left(\frac{z_1^4 - x^4}{r_1^6} + \frac{z_2^4 - x^4}{r_2^6} - \frac{4hz_2(z_2^2 - 3x^2)}{r_2^6}\right) \tag{14}$$

A graphical representation of the isotropic stress σ_0

is shown in Fig. 4. The left half of the figure shows contours of σ_0 for the ground loss problem, and the right half shows contours of σ_0 for the ovalization problem. The thick lines indicate the contours for $\sigma_0 = 0$, and the characters + and − indicate zones of tension and compression respectively. The contour intervals are $0{\cdot}2\mu\varepsilon(R/h)^2$ and $0{\cdot}2\mu\delta(R/h)^2/(1-\nu)$. Although in Fig. 4 it has been assumed that $R/h = 0{\cdot}5$, the contours are valid for any value of R/h. It is interesting to note that in the case of ground loss the stresses around the tunnel are all compressive, whereas in the case of ovalization there are zones of tension above and below the tunnel. The construction of contours of other stress components is a relatively simple and straightforward exercise.

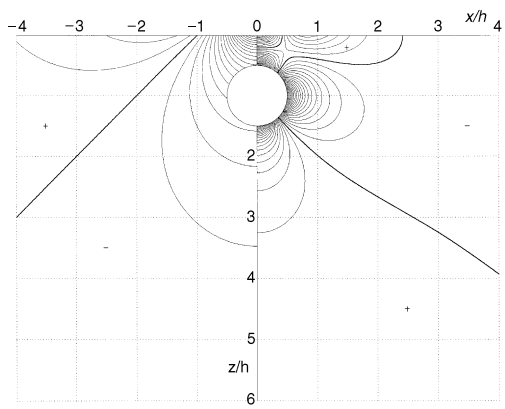

Fig. 4. Contours of isotropic stress

CONCLUSIONS

It appears to be possible to derive approximate analytic solutions for two important cases of tunnel deformation in an elastic soil: a uniform radial displacement due to ground loss, and an ovalization of the tunnel. Simple analytical expressions can be given for the displacements at arbitrary points of the half plane, and for the stress components.

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Erratum

In equations (1) and (2) : $k = \nu/(1-\nu)$.

Surface settlements due to ground loss and ovalization of a tunnel

A. Verruijt and J.R. Booker

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Details of the analysis, and an alternative

The problem

The paper gives an approximate analytic solution for a tunnel in a homogeneous elastic half space, caused by ground loss and ovalization. The two basic processes are illustrated in figure 1. This note presents some additional material and derivations.

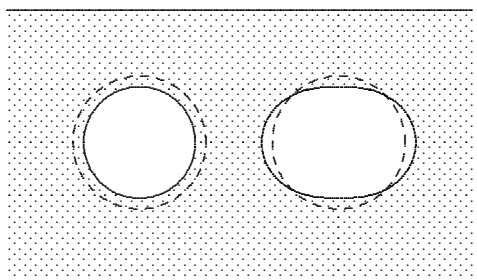

Figure 1: Ground loss and Ovalization of a tunnel.

Basic solution for ground loss

The first solution, describing the ground loss, is a purely radial displacement, decreasing inversely proportional to the distance r from the point $x = 0, y = 0$.

$$u_r = -\frac{\varepsilon R^2}{r}, \tag{1}$$

where u_r is the displacement in radial direction, and ε is a parameter indicating the relative displacement at the circumference of a tunnel of radius R. The total ground loss (per unit length) is $2\varepsilon\pi R^2$, and the relative ground loss, per unit volume of the tunnel, is 2ε. The horizontal and vertical components of the displacement are

$$u_x = -\frac{\varepsilon R^2 x}{r^2}, \tag{2}$$

$$u_y = -\frac{\varepsilon R^2 y}{r^2}, \tag{3}$$

It can easily be verified that these expressions satisfy the basic differential equations for a homogeneous isotropic linear elastic material, deforming under plane strain conditions,

$$m\,\frac{\partial e}{\partial x} + \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} = 0, \tag{4}$$

$$m\,\frac{\partial e}{\partial y} + \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} = 0, \tag{5}$$

where e is the volume strain,

$$e = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}, \tag{6}$$

and the elastic parameter m is defined by

$$m = \frac{1}{1-2\nu}, \tag{7}$$

where ν is Poisson's ratio.

In this case the volume strain appears to be zero,

$$e = 0. \tag{8}$$

The stresses can be determined using Hooke's law for the case of plane strain,

$$\frac{\sigma_{xx}}{2\mu} = \tfrac{1}{2}(m-1)\,e + \frac{\partial u_x}{\partial x} = \frac{\nu}{1-2\nu}\,e + \frac{\partial u_x}{\partial x}, \tag{9}$$

$$\frac{\sigma_{yy}}{2\mu} = \tfrac{1}{2}(m-1)\,e + \frac{\partial u_y}{\partial y} = \frac{\nu}{1-2\nu}\,e + \frac{\partial u_y}{\partial y}, \tag{10}$$

$$\frac{\sigma_{xy}}{2\mu} = \tfrac{1}{2}\Big(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\Big). \tag{11}$$

This gives

$$\frac{\sigma_{xx}}{2\mu} = \frac{\varepsilon R^2}{r^4}(x^2 - y^2), \tag{12}$$

$$\frac{\sigma_{yy}}{2\mu} = \frac{\varepsilon R^2}{r^4}(y^2 - x^2), \tag{13}$$

$$\frac{\sigma_{xy}}{2\mu} = \frac{\varepsilon R^2}{r^4}2xy. \tag{14}$$

Basic solution for ovalization

The basic form of the second type of singularity, describing the ovalization, is

$$u_x = \frac{\delta R^2 x(x^2 - ky^2)}{r^4}, \tag{15}$$

$$u_y = \frac{\delta R^2 y(kx^2 - y^2)}{r^4}, \tag{16}$$

where δ is a measure of the degree of ovalization (the relative increase and decrease of the radius R in the two coordinate directions), and where k is an elastic parameter, defined as

$$k = \frac{\nu}{1-\nu} = \frac{\lambda}{\lambda + 2\mu}. \tag{17}$$

Again it can be verified that this solution satisfies the basic differential equations (4) and (5).

In this case the volume strain is

$$e = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \frac{\delta R^2}{r^6}(k-1)(x^4 - y^4) = -\frac{\delta R^2}{r^6}\frac{1-2\nu}{1-\nu}(x^4 - y^4). \tag{18}$$

The stresses are, with equations (9), (10) and (11),

$$\frac{\sigma_{xx}}{2\mu} = -\frac{\delta R^2}{(1-\nu)r^6}x^2(x^2 - 3y^2), \tag{19}$$

$$\frac{\sigma_{yy}}{2\mu} = -\frac{\delta R^2}{(1-\nu)r^6}y^2(3x^2 - y^2), \tag{20}$$

$$\frac{\sigma_{xy}}{2\mu} = -\frac{\delta R^2}{(1-\nu)r^6}2xy(x^2 - y^2). \tag{21}$$

Solution for two singularities

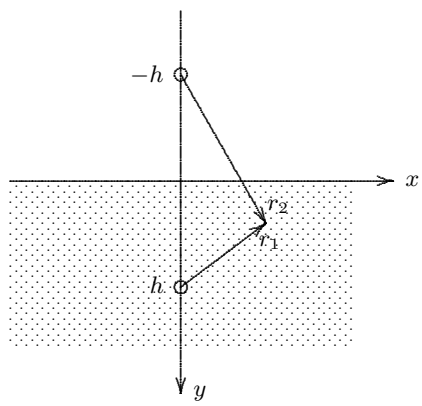

Figure 2: A singularity and its image.

Superposition of two singularities, with ground loss and ovalization, at $y = h$ and at $y = -h$, see figure 2, leads to the displacement field

$$u_x = -\varepsilon R^2\{\frac{x}{r_1^2} + \frac{x}{r_2^2}\} + \delta R^2\{\frac{x(x^2 - ky_1^2)}{r_1^4} + \frac{x(x^2 - ky_2^2)}{r_2^4}\}, \tag{22}$$

$$u_y = -\varepsilon R^2\{\frac{y_1}{r_1^2} + \frac{y_2}{r_2^2}\} + \delta R^2\{\frac{y_1(kx^2 - y_1^2)}{r_1^4} + \frac{y_2(kx^2 - y_2^2)}{r_2^4}\}, \tag{23}$$

where $y_1 = y - h$ and $y_2 = y + h$, and r_1 and r_2 are the distances from the singular point and its image, see figure 2, so that

$$r_1^2 = x^2 + (y-h)^2, \qquad r_2^2 = x^2 + (y+h)^2. \tag{24}$$

The stresses due to the two singularities are

$$\frac{\sigma_{xx}}{2\mu} = \varepsilon R^2\{\frac{x^2 - y_1^2}{r_1^4} + \frac{x^2 - y_2^2}{r_2^4}\} - \frac{\delta R^2}{1-\nu}\{\frac{x^2(x^2 - 3y_1^2)}{r_1^6} + \frac{x^2(x^2 - 3y_2^2)}{r_2^6}\}, \tag{25}$$

$$\frac{\sigma_{yy}}{2\mu} = -\varepsilon R^2\{\frac{x^2 - y_1^2}{r_1^4} + \frac{x^2 - y_2^2}{r_2^4}\} - \frac{\delta R^2}{1-\nu}\{\frac{y_1^2(3x^2 - y_1^2)}{r_1^6} + \frac{y_2^2(3x^2 - y_2^2)}{r_2^6}\}, \tag{26}$$

$$\frac{\sigma_{xy}}{2\mu} = \varepsilon R^2 \{ \frac{2xy_1}{r_1^4} + \frac{2xy_2}{r_2^4} \} - \frac{\delta R^2}{1-\nu} \{ \frac{2xy_1(x^2-y_1^2)}{r_1^6} + \frac{2xy_2(x^2-y_2^2)}{r_2^6} \}. \tag{27}$$

At the surface $y = 0$: $y_1 = -h$ and $y_2 = h$, so that it follows from equations (23) and (27) that on that surface $u_y = 0$ and $\sigma_{xy} = 0$. This is a consequence, of course, of the symmetry of the solution. The normal stresses σ_{yy} induced by the two parts of the solution are equal, and add up to

$$y = 0 \ : \ \sigma_{yy} = q(x) = -4\mu\varepsilon R^2 \frac{x^2-h^2}{(x^2+h^2)^2} - \frac{4\mu\delta R^2}{1-\nu} \frac{h^2(3x^2-h^2)}{(x^2+h^2)^3}, \tag{28}$$

In order to satisfy the boundary condition that the normal stress $\sigma_{yy} = 0$ at the surface $y = 0$, a third solution must be added, which should balance the stress distribution $q(x)$.

The third part of the solution

The third part of the solution is a problem for an elastic half plane with zero shear stresses at the surface $y = 0$, and normal stresses

$$y = 0 \ : \ \sigma_{yy} = -q(x) = 4\mu\varepsilon R^2 \frac{x^2-h^2}{(x^2+h^2)^2} + \frac{4\mu\delta R^2}{1-\nu} \frac{h^2(3x^2-h^2)}{(x^2+h^2)^3}, \tag{29}$$

This is a Boussinesq problem, which can be solved using the Fourier transform method.

The Fourier transform method

The Fourier transform method for solving problems of elasticity for a half plane (Sneddon, 1951) leads to the general type of solution

$$u_x = \int_0^\infty C(1-m\alpha y)\exp(-\alpha y)\sin(\alpha x)\, d\alpha, \tag{30}$$

$$u_y = -\int_0^\infty C(1+m+m\alpha y)\exp(-\alpha y)\cos(\alpha x)\, d\alpha, \tag{31}$$

where, as before, m is defined by

$$m = \frac{1}{1-2\nu}. \tag{32}$$

It can easily be verified that these equations satisfy the basic differential equations (4) and (5).

It follows from equations (30) and (31) that

$$e = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 2\int_0^\infty C\alpha \exp(-\alpha y)\cos(\alpha x)\, d\alpha, \tag{33}$$

$$f = \frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} = -2m\int_0^\infty C\alpha^2 y \exp(-\alpha y)\cos(\alpha x)\, d\alpha, \tag{34}$$

$$g = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 2m\int_0^\infty C\alpha^2 y \exp(-\alpha y)\cos(\alpha x)\, d\alpha. \tag{35}$$

Using these equations the stresses can be expressed as

$$\frac{\sigma_{xx}}{2\mu} = \tfrac{1}{2}me + \tfrac{1}{2}f = m\int_0^\infty C\alpha(1-\alpha y)\exp(-\alpha y)\cos(\alpha x)\, d\alpha, \tag{36}$$

$$\frac{\sigma_{yy}}{2\mu} = \tfrac{1}{2}me - \tfrac{1}{2}f = m\int_0^\infty C\alpha(1+\alpha y)\exp(-\alpha y)\cos(\alpha x)\, d\alpha, \tag{37}$$

$$\frac{\sigma_{xy}}{2\mu} = \tfrac{1}{2}g = m\int_0^\infty C\alpha^2 y \exp(-\alpha y)\cos(\alpha x)\, d\alpha. \tag{38}$$

Ground loss

In the present problem the boundary condition for the stress σ_{yy} is, with equation (29), and considering the ground loss case only,

$$y = 0 \ : \ \frac{\sigma_{yy}}{2\mu} = 2\varepsilon R^2 \frac{x^2 - h^2}{(x^2 + h^2)^2}. \tag{39}$$

With equation (37) this leads to the condition

$$\int_0^\infty C\alpha \exp(-\alpha y) \cos(\alpha x)\, d\alpha = \frac{2\varepsilon R^2}{m} \frac{x^2 - h^2}{(x^2 + h^2)^2}. \tag{40}$$

Using the inversion theorem for the Fourier cosine integral (Sneddon, 1951) it follows that

$$C\alpha = \frac{4\varepsilon R^2}{\pi m} \int_0^\infty \frac{x^2 - h^2}{(x^2 + h^2)^2} \cos(\alpha x)\, dx. \tag{41}$$

Using the Fourier transform (114) from the Appendix gives

$$C = -\frac{2\varepsilon R^2}{m} \exp(-\alpha h). \tag{42}$$

Substitution into the expressions (30) and (31) for the displacements now gives

$$u_x = -\frac{2\varepsilon R^2}{m} \int_0^\infty (1 - m\alpha y) \exp[-\alpha(y + h)] \sin(\alpha x)\, d\alpha, \tag{43}$$

$$u_y = \frac{2\varepsilon R^2}{m} \int_0^\infty (1 + m + m\alpha y) \exp[-\alpha(y + h)] \cos(\alpha x)\, d\alpha. \tag{44}$$

These integrals are all given in the Appendix. The final results are, using the definition of m, see equation (32),

$$u_x = -2\varepsilon R^2 \Big\{ \frac{(1 - 2\nu)x}{r_2^2} - \frac{2xyy_2}{r_2^4} \Big\}, \tag{45}$$

$$u_y = 2\varepsilon R^2 \Big\{ \frac{2(1 - \nu)y_2}{r_2^2} - \frac{y(x^2 - y_2^2)}{r_2^4} \Big\}, \tag{46}$$

where $y_2 = y + h$ and $r_2^2 = x^2 + (y + h)^2$.

With equation (42) the expressions for the stresses become

$$\frac{\sigma_{xx}}{2\mu} = -2\varepsilon R^2 \int_0^\infty \alpha(1 - \alpha y) \exp(-\alpha y_2) \cos(\alpha x)\, d\alpha, \tag{47}$$

$$\frac{\sigma_{yy}}{2\mu} = -2\varepsilon R^2 \int_0^\infty \alpha(1 + \alpha y) \exp(-\alpha y_2) \cos(\alpha x)\, d\alpha, \tag{48}$$

$$\frac{\sigma_{xy}}{2\mu} = -2\varepsilon R^2 \int_0^\infty \alpha^2 y \exp(-\alpha y_2) \cos(\alpha x)\, d\alpha, \tag{49}$$

where $y_2 = y + h$.

All the integrals in these expressions are also given in the Appendix. This gives

$$\frac{\sigma_{xx}}{2\mu} = 2\varepsilon R^2 \Big\{ \frac{x^2 - y_2^2}{r_2^4} - \frac{2yy_2(3x^2 - y_2^2)}{r_2^6} \Big\}, \tag{50}$$

$$\frac{\sigma_{yy}}{2\mu} = 2\varepsilon R^2 \Big\{ \frac{x^2 - y_2^2}{r_2^4} + \frac{2yy_2(3x^2 - y_2^2)}{r_2^6} \Big\}, \tag{51}$$

$$\frac{\sigma_{xy}}{2\mu} = 2\varepsilon R^2 \frac{2xy(x^2 - 3y_2^2)}{r_2^6}. \tag{52}$$

Ovalization

For the ovalization case the boundary condition for the stress σ_{yy} is, with equation (29) and $\varepsilon = 0$,

$$y = 0 \ : \ \frac{\sigma_{yy}}{2\mu} = \frac{2\delta R^2}{1-\nu}\,\frac{h^2(3x^2-h^2)}{(x^2+h^2)^3}, \tag{53}$$

With equation (37) this leads to the condition

$$\int_0^\infty C\alpha\exp(-\alpha y)\cos(\alpha x)\,d\alpha = 2\delta R^2\,\frac{1-2\nu}{1-\nu}\,\frac{h^2(3x^2-h^2)}{(x^2+h^2)^3}. \tag{54}$$

Using the inversion theorem for the Fourier cosine integral (Sneddon, 1951) it follows that

$$C\alpha = \frac{4\delta R^2}{\pi}\,\frac{1-2\nu}{1-\nu}\int_0^\infty \frac{h^2(3x^2-h^2)}{(x^2+h^2)^3}\cos(\alpha x)\,dx. \tag{55}$$

This integral is determined in the Appendix, see equation (116). In this case the result is

$$C = -\frac{2\delta R^2}{m+1}\alpha h\exp(-\alpha h), \tag{56}$$

in which the factor $2/(m+1)$ can also be written as $(1-2\nu)/(1-\nu)$.

Substitution of (56) into the expressions (30) and (31) for the displacements now gives

$$u_x = -\frac{2\delta R^2}{m+1}\int_0^\infty \alpha h(1-m\alpha y)\exp[-\alpha(y+h)]\sin(\alpha x)\,d\alpha, \tag{57}$$

$$u_y = \frac{2\delta R^2}{m+1}\int_0^\infty \alpha h(1+m+m\alpha y)\exp[-\alpha(y+h)]\cos(\alpha x)\,d\alpha. \tag{58}$$

The Fourier integrals in these expressions are given in the Appendix. This gives

$$u_x = -\frac{4\delta R^2 xh}{m+1}\Big\{\frac{y_2}{r_2^4} + \frac{my(x^2-3y_2)^2}{r_2^6}\Big\}, \tag{59}$$

$$u_y = -2\delta R^2 h\Big\{\frac{x^2-y_2^2}{r_2^4} + \frac{2m}{m+1}\,\frac{yy_2(3x^2-y_2^2)}{r_2^6}\Big\}. \tag{60}$$

With equation (42) the expressions for the stresses become

$$\frac{\sigma_{xx}}{2\mu} = -\frac{2\delta R^2 m}{m+1}\int_0^\infty \alpha^2 h(1-\alpha y)\exp(-\alpha y_2)\cos(\alpha x)\,d\alpha, \tag{61}$$

$$\frac{\sigma_{yy}}{2\mu} = -\frac{2\delta R^2 m}{m+1}\int_0^\infty \alpha^2 h(1+\alpha y)\exp(-\alpha y_2)\cos(\alpha x)\,d\alpha, \tag{62}$$

$$\frac{\sigma_{xy}}{2\mu} = -\frac{2\delta R^2 m}{m+1}\int_0^\infty \alpha^3 hy\exp(-\alpha y_2)\cos(\alpha x)\,d\alpha, \tag{63}$$

where $y_2 = y + h$.

All the integrals in these expressions are also given in the Appendix. This gives

$$\frac{\sigma_{xx}}{2\mu} = \frac{2\delta R^2 h}{1-\nu}\Big\{\frac{y_2(3x^2-y_2^2)}{r_2^6} + \frac{3y(x^4-6x^2y_2^2+y_2^4)}{r_2^8}\Big\}, \tag{64}$$

$$\frac{\sigma_{yy}}{2\mu} = \frac{2\delta R^2 h}{1-\nu}\Big\{\frac{y_2(3x^2-y_2^2)}{r_2^6} - \frac{3y(x^4-6x^2y_2^2+y_2^4)}{r_2^8}\Big\}, \tag{65}$$

$$\frac{\sigma_{xy}}{2\mu} = \frac{24\delta R^2 h}{1-\nu}\,\frac{xyy_2(x^2-y_2^2)}{r_2^8}. \tag{66}$$

An alternative solution for ovalization

It has been observed by Strack (2002) that in the solution for the ovalization case presented in the paper considered above, the tangential displacement u_t does not vanish at the boundary of the cavity, in contrast with his complex variable solution. For the purpose of comparison an alternative may be used, in which $u_t = 0$ along the cavity boundary. This alternative is

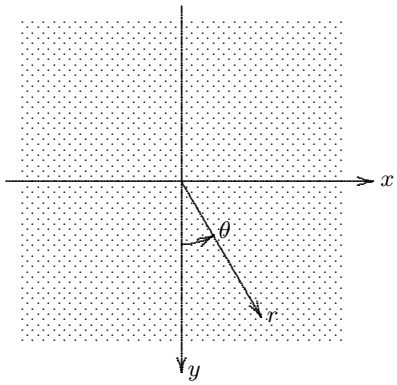

Figure 3: Cartesian and polar coordinates

presented here.

The general solution for ovalization of a circular cavity in an infinite elastic plane is, in polar coordinates, see figure 3,

$$u_r = \{\frac{BR^2(m+1)}{mr} + \frac{DR^4}{3r^3}\}\cos 2\theta, \tag{67}$$

$$u_t = \{-\frac{BR^2}{mr} + \frac{DR^4}{3r^3}\}\sin 2\theta, \tag{68}$$

where, as before, $m = 1/(1-2\nu)$.

The condition that $u_t = 0$ for $r = R$ gives

$$D = \frac{3B}{m}. \tag{69}$$

The solution now reduces to

$$u_r = \frac{BR^2}{mr}\{(m+1) + \frac{R^2}{r^2}\}\cos 2\theta, \tag{70}$$

$$u_t = \frac{BR^2}{mr}\{-1 + \frac{R^2}{r^2}\}\sin 2\theta. \tag{71}$$

The relation between polar coordinates and cartesian coordinates is

$$u_x = u_r \sin\theta + u_t \cos\theta = \frac{x}{r}u_r + \frac{y}{r}u_t, \tag{72}$$

$$u_y = u_r \cos\theta - u_t \sin\theta = \frac{y}{r}u_r - \frac{x}{r}u_t. \tag{73}$$

With

$$\frac{BR}{m} = -\frac{\delta R}{m+2}, \tag{74}$$

one obtains

$$u_x = \frac{\delta R^2 x}{(m+2)r^2}\Big\{1 - m\frac{y^2-x^2}{r^2} - \frac{R^2(3y^2-x^2)}{r^4}\Big\}, \tag{75}$$

$$u_y = -\frac{\delta R^2 y}{(m+2)r^2}\Big\{1 + m\frac{y^2-x^2}{r^2} + \frac{R^2(y^2-3x^2)}{r^4}\Big\}. \tag{76}$$

It can easily be seen that

$$x = R,\ y = 0\ :\ u_x = \delta R,\ u_y = 0, \qquad x = -R,\ y = 0\ :\ u_x = -\delta R,\ u_y = 0, \tag{77}$$

$$x = 0,\ y = R\ :\ u_x = 0,\ u_y = -\delta R, \qquad x = 0,\ y = -R\ :\ u_x = 0,\ u_y = \delta R, \tag{78}$$

which is in agreement with the assumed ovalization of the circle of radius R.

It follows from equations (75) and (76) that

$$\frac{\partial u_x}{\partial x} = \frac{\delta R^2}{(m+2)r^2}\Big\{\frac{y^2-x^2}{r^2} - m\frac{x^4-6x^2y^2+y^4}{r^4} - \frac{3R^2(x^4-6x^2y^2+y^4)}{r^6}\Big\}, \tag{79}$$

$$\frac{\partial u_x}{\partial y} = \frac{2\delta R^2 xy}{(m+2)r^4}\Big\{-1 - m\frac{3x^2-y^2}{r^2} - \frac{6R^2(x^2-y^2]}{r^4}\Big\}, \tag{80}$$

$$\frac{\partial u_y}{\partial x} = \frac{2\delta R^2 xy}{(m+2)r^4}\Big\{1 - m\frac{x^2-3y^2}{r^2} - \frac{6R^2(x^2-y^2)}{r^4}\Big\}. \tag{81}$$

$$\frac{\partial u_y}{\partial y} = \frac{\delta R^2}{(m+2)r^2}\Big\{\frac{y^2-x^2}{r^2} + m\frac{x^4-6x^2y^2+y^4}{r^4} + \frac{3R^2(x^4-6x^2y^2+y^4)}{r^6}\Big\}. \tag{82}$$

It now follows that

$$e = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \frac{2\delta R^2}{(m+2)r^2}\Big\{\frac{y^4-x^4}{r^4}\Big\} \tag{83}$$

$$f = \frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} = -\frac{2\delta R^2}{(m+2)r^2}\Big\{m\frac{x^4-6x^2y^2+y^4}{r^4} + \frac{3R^2(x^4-6x^2y^2+y^4)}{r^6}\Big\} \tag{84}$$

$$g = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -\frac{8\delta R^2}{(m+2)r^2}\Big\{\frac{mxy(x^2-y^2)}{r^4} + \frac{3R^2xy(x^2-y^2)}{r^6}\Big\}. \tag{85}$$

Using these relations the stresses are found to be

$$\frac{\sigma_{xx}}{2\mu} = \tfrac{1}{2}(me+f) = -\frac{\delta R^2}{(m+2)r^2}\Big\{m\frac{2x^2(x^2-3y^2)}{r^4} + \frac{3R^2(x^4-6x^2y^2+y^4}{r^6}\Big\}, \tag{86}$$

$$\frac{\sigma_{yy}}{2\mu} = \tfrac{1}{2}(me-f) = -\frac{\delta R^2}{(m+2)r^2}\Big\{m\frac{2y^2(3x^2-y^2)}{r^4} - \frac{3R^2(x^4-6x^2y^2+y^4}{r^6}\Big\}, \tag{87}$$

$$\frac{\sigma_{xy}}{2\mu} = \tfrac{1}{2}g = -\frac{4\delta R^2}{(m+2)r^2}\Big\{\frac{mxy(x^2-y^2)}{r^4} + \frac{3R^2xy(x^2-y^2)}{r^6}\Big\}. \tag{88}$$

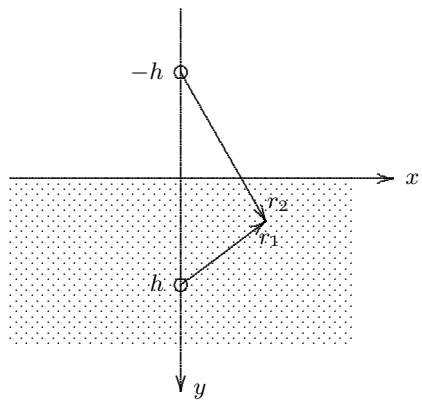

Figure 4: A singularity and its image.

Solution for two singularities

Superposition of two of these ovalization singularities, at $y = h$ and at $y = -h$, see figure 4, leads to the displacement field

$$u_x = \frac{\delta R^2 x}{m+2}\Big\{\frac{1}{r_1^2} - m\frac{y_1^2 - x^2}{r_1^4} - \frac{R^2(3y_1^2 - x^2)}{r_1^6} + \frac{1}{r_2^2} - m\frac{y_2^2 - x^2}{r_2^4} - \frac{R^2(3y_2^2 - x^2)}{r_2^6}\Big\}, \tag{89}$$

$$u_y = -\frac{\delta R^2}{m+2}\Big\{\frac{y_1}{r_1^2} + m\frac{y_1(y_1^2 - x^2)}{r_1^4} + \frac{R^2 y_1(y_1^2 - 3x^2)}{r_1^6} + \frac{y_2}{r_2^2} + m\frac{y_2(y_2^2 - x^2)}{r_2^4} + \frac{R^2 y_2(y_2^2 - 3x^2)}{r_2^6}\Big\}, \tag{90}$$

where $y_1 = y - h$, $y_2 = y + h$, $r_1^2 = x^2 + y_1^2$ and $r_2^2 = x^2 + y_2^2$.

The stress field for these two singularities is

$$\frac{\sigma_{xx}}{2\mu} = -\frac{\delta R^2}{m+2}\Big\{m\frac{2x^2(x^2 - 3y_1^2)}{r_1^6} + \frac{3R^2(x^4 - 6x^2y_1^2 + y_1^4)}{r_1^8} + m\frac{2x^2(x^2 - 3y_2^2)}{r_2^6} + \frac{3R^2(x^4 - 6x^2y_2^2 + y_2^4)}{r_2^8}\Big\}, \tag{91}$$

$$\frac{\sigma_{yy}}{2\mu} = -\frac{\delta R^2}{m+2}\Big\{m\frac{2y_1^2(3x^2 - y_1^2)}{r_1^6} - \frac{3R^2(x^4 - 6x^2y_1^2 + y_1^4)}{r_1^8} + m\frac{2y_2^2(3x^2 - y_2^2)}{r_2^6} - \frac{3R^2(x^4 - 6x^2y_2^2 + y_2^4)}{r_2^8}\Big\}, \tag{92}$$

$$\frac{\sigma_{xy}}{2\mu} = -\frac{4\delta R^2}{m+2}\Big\{\frac{mxy_1(x^2 - y_1^2)}{r_1^6} + \frac{3R^2xy_1(x^2 - y_1^2)}{r_1^8} + \frac{mxy_2(x^2 - y_2^2)}{r_2^6} + \frac{3R^2xy_2(x^2 - y_2^2)}{r_2^8}\Big\}. \tag{93}$$

At the free surface $y = 0$, so that $y_1 = -h$, $y_2 = h$ and $r_1^2 = r_2^2 = x^2 + h^2$. It then follows that the vertical displacement u_y and the shear stress σ_{xy} on that surface vanish,

$$y = 0 \ : \ u_y = 0, \tag{94}$$

$$y = 0 \ : \ \sigma_{xy} = 0. \tag{95}$$

This is a consequence of the symmetry of the problem, of course.

The vertical normal stress on the surface $y = 0$ is found to be

$$y = 0 \ : \ \sigma_{yy} = 2\mu f(x), \tag{96}$$

where now

$$f(x) = \frac{2\delta R^2}{m+2}\Big\{m\frac{2h^2(h^2-3x^2)}{(h^2+x^2)^3} + \frac{3R^2(x^4-6x^2h^2+h^4)}{(h^2+x^2)^4}\Big\}. \tag{97}$$

This can also be written as

$$f(x) = \frac{2\delta R^2}{m+2}\Big\{-6m\frac{h^2}{(h^2+x^2)^2} + 8m\frac{h^4}{(h^2+x^2)^3} + \frac{3R^2}{(h^2+x^2)^2} - \frac{24R^2h^2}{(h^2+x^2)^3} + \frac{24R^2h^4}{(h^2+x^2)^4}\Big\}. \tag{98}$$

Written in this form the numerators no longer contain terms dependent upon x. This makes the Fourier integrals somewhat easier to determine.

The third part of the solution

The third part of the solution is a problem for an elastic half plane with zero shear stresses at the surface $y = 0$, and normal stresses

$$y = 0 \ : \ \sigma_{yy} = -2\mu f(x). \tag{99}$$

This is a Boussinesq problem, which can be solved using the Fourier transform method.

With equation (37) this leads to the condition

$$m\int_0^\infty C\alpha\cos(\alpha x)\,d\alpha = -f(x). \tag{100}$$

Application of the inversion theorem for the Fourier cosine integral (Sneddon, 1951) now gives

$$C\alpha = -\frac{2}{m\pi}\int_0^\infty f(x)\cos(\alpha x)\,d\alpha. \tag{101}$$

Using some of the Fourier integrals derived in the Appendix the value of C is found to be

$$C = -\frac{2\delta R^2}{m+2}\Big\{\alpha h + +\frac{R^2}{2mh^2}(\alpha h)^2]\Big\}\exp(-\alpha h). \tag{102}$$

Although this form may be used to derive the displacements and stresses for the third part of the solution, it is assumed here that the radius of the tunnel is small, $R/h \ll 1$. In that case equation (102) reduces to

$$C = -\frac{2\delta R^2}{m+2}\alpha h\exp(-\alpha h). \tag{103}$$

This is the same solution as obtained in the original problem, see equation (56), except that the factor $m+1$ in the denominator must be replaced by $m+2$.

It follows that the additional expressions for the displacements in this case are

$$u_x = -\frac{4\delta R^2 x h}{m+2}\left\{\frac{y_2}{r_2^4} + \frac{my(x^2-3y_2)^2}{r_2^6}\right\}, \tag{104}$$

$$u_y = -\frac{2\delta R^2 h}{m+2}\left\{(m+1)\,\frac{x^2-y_2^2}{r_2^4} + 2m\,\frac{yy_2(3x^2-y_2^2)}{r_2^6}\right\}. \tag{105}$$

And the expressions for the stresses are

$$\frac{\sigma_{xx}}{2\mu} = \frac{4m\delta R^2 h}{m+2}\left\{\frac{y_2(3x^2-y_2^2)}{r_2^6} + \frac{3y(x^4-6x^2y_2^2+y_2^4)}{r_2^8}\right\}, \tag{106}$$

$$\frac{\sigma_{yy}}{2\mu} = \frac{4m\delta R^2 h}{m+2}\left\{\frac{y_2(3x^2-y_2^2)}{r_2^6} - \frac{3y(x^4-6x^2y_2^2+y_2^4)}{r_2^8}\right\}, \tag{107}$$

$$\frac{\sigma_{xy}}{2\mu} = \frac{48m\delta R^2 h}{m+2}\,\frac{xyy_2(x^2-y_2^2)}{r_2^8}. \tag{108}$$

Reference

O.E. Strack, *Analytic Solutions of Elastic Tunneling Problems*, Ph.D. Thesis, Delft, 2002.

APPENDIX : SOME INTEGRAL TRANSFORMS

In this appendix a number of integral transforms are assembled, mainly from the literature. Many of these integrals are used in this paper.

A well known Laplace transform (Churchill, 1972), which can easily be derived by using partial differentiation, is

$$\int_0^\infty \cos(at)\exp(-st)\,dt = \frac{s}{s^2+a^2}. \tag{109}$$

By a simple change of notation this integral can also be written as a Fourier cosine transform,

$$\int_0^\infty \exp(-\alpha h)\cos(\alpha x)\,d\alpha = \frac{h}{h^2+x^2}. \tag{110}$$

Using the Fourier inversion theorem (Sneddon, 1951), it now follows that

$$\frac{2}{\pi}\int_0^\infty \frac{h}{h^2+x^2}\cos(\alpha x)\,dx = \exp(-\alpha h), \tag{111}$$

or

$$\int_0^\infty \frac{1}{h^2+x^2}\cos(\alpha x)\,dx = \frac{\pi}{2h}\exp(-\alpha h). \tag{112}$$

Differentiation of (112) with respect to h gives, after division by $-2h$,

$$\int_0^\infty \frac{1}{(h^2+x^2)^2}\cos(\alpha x)\,dx = \frac{\pi}{4h^3}(1+\alpha h)\exp(-\alpha h). \tag{113}$$

Because $h^2 - x^2 = 2h^2 - (h^2+x^2)$ it follows from (112) and (113) that

$$\int_0^\infty \frac{h^2-x^2}{(h^2+x^2)^2}\cos(\alpha x)\,dx = \frac{\pi\alpha}{2}\exp(-\alpha h). \tag{114}$$

Differentiation of (113) with respect to h gives, after division by $-4h$,

$$\int_0^\infty \frac{1}{(h^2+x^2)^3}\cos(\alpha x)\,dx = \frac{\pi}{16h^5}[3+3\alpha h+(\alpha h)^2]\exp(-\alpha h). \tag{115}$$

Because $h^2 - 3x^2 = 4h^2 - 3(h^2+x^2)$ it follows from (113) and (115) that

$$\int_0^\infty \frac{h^2-3x^2}{(h^2+x^2)^2}\cos(\alpha x)\,dx = \frac{\pi\alpha^2}{4h}\exp(-\alpha h). \tag{116}$$

Differentiation of (115) with respect to h gives, after division by $-6h$,

$$\int_0^\infty \frac{1}{(h^2+x^2)^4}\cos(\alpha x)\,dx = \frac{\pi}{96h^7}[15+15\alpha h+6(\alpha h)^2+(\alpha h)^3]\exp(-\alpha h). \tag{117}$$

Now returning to the second integral, eq. (110), it follows, by differentiation with respect to h, that

$$\int_0^\infty \alpha\cos(\alpha x)\exp(-\alpha h)\,d\alpha = \frac{h^2-x^2}{(h^2+x^2)^2}. \tag{118}$$

Differentiating this equation once more with respect to h gives

$$\int_0^\infty \alpha^2 \cos(\alpha x) \exp(-\alpha h)\, d\alpha = \frac{2h(h^2 - 3x^2)}{(h^2 + x^2)^3}. \tag{119}$$

It may be noted that (118) and (119) are the inverse Fourier transforms of (114) and (116). Differentiating equation (119 with respect to h gives

$$\int_0^\infty \alpha^3 \cos(\alpha x) \exp(-\alpha h)\, d\alpha = \frac{6(h^4 - 6h^2x^2 + x^4)}{(h^2 + x^2)^4}. \tag{120}$$

Another well known Laplace transform is

$$\int_0^\infty \sin(at) \exp(-st)\, dt = \frac{a}{s^2 + a^2}. \tag{121}$$

Written in the notation of a Fourier sine transform this integral is

$$\int_0^\infty \sin(\alpha x) \exp(-\alpha h)\, d\alpha = \frac{x}{h^2 + x^2}. \tag{122}$$

Differentiation with respect to h gives

$$\int_0^\infty \alpha \sin(\alpha x) \exp(-\alpha h)\, d\alpha = \frac{2hx}{(h^2 + x^2)^2}. \tag{123}$$

Repeating the differentiation with respect to h gives

$$\int_0^\infty \alpha^2 \sin(\alpha x) \exp(-\alpha h)\, d\alpha = \frac{2x(3h^2 - x^2)}{(h^2 + x^2)^3}. \tag{124}$$

Another differentiation with respect to h gives

$$\int_0^\infty \alpha^3 \sin(\alpha x) \exp(-\alpha h)\, d\alpha = \frac{24xh(h^2 - x^2)}{(h^2 + x^2)^4}. \tag{125}$$